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## Scattering of Scalar Field by an Extended Black Hole in F(R) gravity

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In this work we have studied the scattering of scalar field around an extended black hole in F(R) gravity using WKB method. We have obtained the wave function in different regions such as near the horizon region, away from horizon and far away from horizon and the absorption cross section are calculated. We find that the absorption cross section is inversely proportional to the cube of Hawking temperature. We have also evaluated the Hawking temperature of the black hole via tunneling method.

*Keywords:* Black hole, F(R) theory, Extended Black hole solution , scattering, absorption cross section, Hawking temperature.

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### 1. Introduction

The late time acceleration of the universe<sup>1,2</sup> introduced new problems in gravitational physics. Modification of Einstein's general theory of relativity is required to explain the late time cosmic acceleration. There are two methods for explaining these new observations, one method is to introduce the concept of dark energy to provide the necessary force for acceleration and the other method is to modify Einstein's equation of General Theory of Relativity(GTR). F(R) model gravity is an attempt to modify GTR<sup>3</sup>, for a review see reference [4]<sup>4</sup>. There exist black hole solutions in this new formulation of gravity also<sup>5,6</sup>.

There are a number of modified gravity theories and F(R) theory is the simplest one. In Einstein's formulation the Ricci scalar is used as the Lagrangian density. In F(R) theory, we use a general function of Ricci scalar as the Lagrangian density. Due

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to the appearance of the function of the Ricci scalar the resulting field equation is fourth order, while the field equation in Einstein's general relativity is second order. This makes the study of  $F(R)$  field equation more difficult but attempts have been made to obtain cosmological solution and Black hole solution in this new frame work.

Black holes are objects capable of absorbing all particles and radiation that enter the event horizon. But in 1970s Bekenstein<sup>7</sup> and Hawking<sup>8</sup> developed the black hole thermodynamics which indicates that Black Holes have a characteristic temperature and entropy. Semi-classical theory shows that Black Holes can emit particles<sup>9</sup>. This is called particle creation by Black Holes. These studies led to a new area of study called thermodynamics of black holes and this area has attracted attention of many. Classically all matter entering the event horizon get absorbed by the black hole. But quantum mechanically there is a probability of matter and/or fields getting scattered. This shows a non-zero reflectivity at the horizon. Using this assumption a number of works have appeared in the literature studying Black Hole scattering<sup>10</sup>.

The paper is organized as follows. In section 2 we give the basic theory of black hole scattering assuming that black hole is surrounded by a scalar field which obeys the Klein-Gordon equation. In section 3 we study the the wave function of scattering field in the vicinity of the horizon. The solution of wave equation in a region  $r$  greater than  $r_h$  is studied in the section 4. In section 5 the solution at far away from horizon is studied. The absorption cross section is studied in section 6. In section 7 we calculate the Hawking temperature via tunneling method. Conclusion of the present study is given in section 8.

## 2. Theory of black hole scattering

The action for the gravity in  $F(R)$  theory can be written as

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(R), \quad (1)$$

where  $g$  is the determinant of the metric and  $F$  is a general function of the Ricci scalar  $R$ . We start with a static spherically symmetric solution of the form<sup>11</sup>

$$ds^2 = -e^{2\beta(r)} B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2. \quad (2)$$

Following reference.[11], we assume that  $\beta$  is constant and  $F(R) = \sqrt{R + 6C_2}$ . We get a solution of the form

$$-B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2, \quad (3)$$

with  $B(r)$  equal to,

$$B(r) = 1 - \frac{c_1}{r^2} + c_2 r^2. \quad (4)$$

The  $c_2$  term in the solution represents a cosmological constant term. Assuming the cosmological constant term is zero ie.  $c_2 = 0$  and  $c_1 = 2\alpha m$ , we get,

$$ds^2 = -(1 - \frac{2m\alpha}{r^2})dt^2 + \frac{dr^2}{(1 - \frac{2m\alpha}{r^2})} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \quad (5)$$

where all the symbols except  $\alpha$  have the usual meaning.  $\alpha$  is a constant parameter having the dimension of length. The metric has singularities at both  $r = \sqrt{2\alpha m}$  and  $r = 0$ . Since  $\alpha$  is a constant parameter forms a black hole solution with horizon at  $r = \sqrt{2\alpha m}$ . We will now invert the metric into tortoise co-ordinate as

$$ct = \pm r \pm \sqrt{2\alpha m} \operatorname{arctanh}\left[\frac{r}{\sqrt{2\alpha m}}\right] + \text{constant}, \quad (6)$$

where the minus sign is for ingoing photon and positive sign is for outgoing photon. Proceeding as in GTR, we first obtain a new coordinate  $p$  with  $p$  as

$$ct = -r - \sqrt{2\alpha m} \operatorname{arctanh}\left[\frac{r}{\sqrt{2\alpha m}}\right] + p. \quad (7)$$

With the null coordinate  $p$ , the metric simplyfy Eq.(5) can be written as

$$ds^2 = \left(1 - \frac{2\alpha m}{r^2}\right) dp^2 - 2dpdr - r^2 d\Omega^2. \quad (8)$$

Now defining a time like coordinate  $t'$  as  $ct' = p - r$  so that we get the Eddington-Finkelstein coordinate metric as

$$ds^2 = c^2 \left(1 - \frac{2\alpha m}{r^2}\right) dt'^2 - \frac{4m\alpha c}{r^2} dt' dr - \left(1 + \frac{2\alpha m}{r^2}\right) dr^2 - r^2 d\Omega^2 \quad (9)$$

The Eddington-Finkelstein coordinate metric for the extended black hole has the same form as Schwarzschild metric and thus this metric can have a regular null surface at  $r = \sqrt{2\alpha m}$ .

We now place the Black hole immersed in a massive scalar field, the field equation in this back ground can be described by the Klein-Gordon equation<sup>13</sup>

$$(\square + \mu^2)\Phi = 0. \quad (10)$$

In curved space-time this equation can be written in the following form,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) + \mu^2 \Phi = 0. \quad (11)$$

Writing this equation in the spherical polar coordinate we find<sup>12</sup>

$$\frac{1}{r^2(\sin\theta)} \left[ (\partial_t(r^2(\sin\theta)) \frac{1}{(1 - \frac{2m\alpha}{r^2})} \partial_t) - (\partial_r r^2 \sin\theta (1 - \frac{2m\alpha}{r^2}) \partial_r) - (\partial_\theta r^2 \sin\theta \frac{1}{r^2} \partial_\theta) - (\frac{1}{\sin\theta} \partial_\phi^2) \right] \Phi = 0. \quad (12)$$

In order to separate this equation into radial and the angular parts, we assume that  $\Phi$  is of the form

$$\Phi(r, t) = \exp(-i\epsilon t)\phi_l(r)Y_{lm}(\theta, \phi), \quad (13)$$

where  $\epsilon$  is the energy,  $l$  and  $m$  are the angular momentum and its projection respectively. We consider only the radial part and it takes the following form

$$\left[\frac{1}{r^2}\partial_r(r^2 - 2\alpha m)\partial_r + \frac{r^2\epsilon^2}{(r + \sqrt{2\alpha m})(r - \sqrt{2\alpha m})} - \frac{l(l+1)}{r^2} + \mu^2\right]\phi_l(r) = 0. \quad (14)$$

### 3. Solution of wave equation in the vicinity of horizon

We use WKB approximation for solving the radial scattering equation and assume<sup>14</sup> a solution of the form,

$$\phi_l(r) = \exp(-i \int k(r) dr). \quad (15)$$

Using Eq. 15 in Eq.7 we get,

$$\left[\frac{\epsilon^2}{1 - \frac{2\alpha m}{r^2}} - \frac{l(l+1)}{r^2} + \mu^2\right] + [-k^2(1 - \frac{2\alpha m}{r^2})] = 0. \quad (16)$$

On rearranging we find,

$$k^2 = (1 - \frac{2\alpha m}{r^2})^{-2}[\epsilon^2 - (\frac{l(l+1)}{r^2} + \mu^2)(1 - \frac{2\alpha m}{r^2})]. \quad (17)$$

Since we are considering a situation where  $r$  approaches  $r_h$  and hence we need only consider the  $\epsilon^2$  term in the square bracket. Then  $k$  is given by,

$$k = (1 - \frac{2\alpha m}{r^2})^{-1}\epsilon. \quad (18)$$

and

$$(1 - \frac{2\alpha m}{r^2}) = \frac{1}{r^2}(r^2 - 2\alpha m) = \frac{1}{r^2}(r + \sqrt{2\alpha m})(r - \sqrt{2\alpha m}) \quad (19)$$

When  $r \rightarrow r_h$ ,  $k$  can be written in a compact form as,

$$k = \frac{\xi}{(r - r_h)}. \quad (20)$$

where  $\xi$  is

$$\xi = \frac{r_h^2\epsilon}{r + r_h}. \quad (21)$$

Hence from our above assumption, the wave function as  $r \rightarrow r_h$  becomes

$$\phi_l = e^{i \int \frac{\xi}{r - r_h} dr}. \quad (22)$$

which can be integrated to the following form,

$$\phi_l = e^{i\xi \ln(r - r_h)}. \quad (23)$$

Thus the wave function in the vicinity of the event horizon may be written as,

$$\phi_l = e^{\pm i\xi \ln(r-r_h)}. \quad (24)$$

Now we will consider the scalar wave approaching the black hole horizon and using Eq. (24) the wave function in the vicinity of the black hole horizon can be written, assuming the field gets reflected at the black hole horizon, as

$$\phi_l = e^{-i\xi \ln(r-r_h)} + |R|e^{+i\xi \ln(r-r_h)} \quad (25)$$

where  $R$  represents the reflection coefficient. If  $R \neq 0$ , there is a definite probability for the incident wave to get reflected at the horizon.

#### 4. Solution of wave equation in a region $r$ greater than $r_h$ .

In this section we consider a situation where the field is sufficiently away from the event horizon. We also assume a situation that the energy and momentum,  $\epsilon$  and  $\mu$  of the field are very small and they can be neglected. For s-wave Eq. (14) now takes the form,

$$\frac{1}{r^2} \partial_r(r^2 - 2\alpha m \partial_r \phi_0) = 0. \quad (26)$$

From which we obtain, after some calculations, the following equation

$$\ln \phi'_0(r) = -\ln(2r(r + \sqrt{2\alpha m})(r - \sqrt{2\alpha m})), \quad (27)$$

and the wave function is obtained as,

$$\phi_0 = C \ln\left(\frac{r-r_h}{r}\right). \quad (28)$$

Comparing the solutions for regions 1 and 2, we have in Region 1 the wave function as

$$\phi_l = e^{i\xi \ln(r-r_h)}, \quad (29)$$

which contains both incoming and outgoing waves. For  $s$ -wave, we can rewrite the solution for Region 1, as

$$\phi_0 = 1 - i\xi \ln(r-r_h) + R(1 + i\xi \ln(r-r_h)). \quad (30)$$

Where  $R$  is the reflection coefficient<sup>15</sup>. This can be further simplified as

$$\phi_0 = -i\xi \ln(r-r_h)(1-R) + (1+R). \quad (31)$$

Thus in Region 2 also, we can write the  $s$ -wave solution as

$$\phi_0 = \alpha \ln \frac{r-r_h}{r} + \beta, \quad (32)$$

where  $\alpha = i\xi(1-R)$  and  $\beta = (1+R)$ .

In the next section we will study the behaviour scalar field in a region sufficiently far away from the horizon.

### 5. Solution of wave equation far away from the horizon

In this region, the radial part of the wave equation can be written as

$$\phi_l'' + \frac{2}{r}\phi_l' + p^2\phi_l = 0, \quad (33)$$

with  $p$  is the linear momentum associated with scalar field and is  $p^2 = \epsilon^2 - \mu^2$ .

Using Frobenius method we can solve the above equation. The solution is given by

$$\phi_l = \frac{1}{r}(A_l e^{iz} + B_l e^{-iz}), \quad (34)$$

with  $z = pr$ . This can be again simplified as

$$\phi_l = \frac{1}{r}(aF(r) + bG(r)), \quad (35)$$

where  $F(r) = \sin(pr)$  and  $G(r) = \cos(pr)$ . At the boundary between Region 2 and Region 3, where  $pr \ll 1$  then we can expand both sine and cosine terms and need to take only first terms; then  $F(r) \approx pr$  and  $G(r) \approx 1$  and the  $s$  - wave solution is given by,

$$\phi_0 = ap + \frac{b}{r}, \quad (36)$$

and in Region 2,  $\phi_0$  is given by

$$\phi_0 = -\alpha \frac{r_h}{r} + \beta \quad (37)$$

From region 1 we get the wave function finally as

$$\phi_0 = i\xi(1 - R)\frac{r_h}{r} + (1 + R). \quad (38)$$

Comparing equations (36) and (37) we can obtain  $a$  and  $b$  as  $a = \frac{1+R}{p}$ ,  $b = i\xi r_h(1 - R)$ . Now we will calculate the absorption coefficient, assuming the wave gets reflected at the horizon.

### 6. Absorption cross section

The absorption cross section can be evaluated using the above data. We now consider the solution of Eq.(32) given by Eq.(33). The scattering matrix element is defined as

$$S_l = (-1)^{l+1} \frac{A_l}{B_l} e^{2i\delta_l}, \quad (39)$$

where  $\delta_l$  is the phase shift corresponding to angular momentum  $l$ .

We take the low energy limit so that  $l = 0$ . Using Eqs. (33), (35) and (37), we can obtain, for  $s$  - waves,  $A_0 = \frac{a+ib}{2i}$  and  $B_0 = \frac{-a+ib}{2i}$ . Substituting the values of  $a$  and  $b$  we get  $S_0$  as

$$S_0 = \frac{(1 + R) - \xi pr_h(1 - R)}{(1 + R) + \xi pr_h(1 - R)}. \quad (40)$$

Defining  $\eta = \frac{1-R}{1+R}$  the equation of  $S_0$  takes the form

$$S_0 = \frac{1 - \xi p r_h \eta}{1 + \xi p r_h \eta}. \quad (41)$$

The absorption cross section  $\sigma_{abs}$ , is then given by

$$\sigma_{abs} = \frac{\pi}{p^2} (1 - S_0^2). \quad (42)$$

Using the above equations and also using the relation  $p = \epsilon v$  we obtain finally

$$\sigma_{abs} = \frac{2\pi^2 \epsilon r_h^3}{v}. \quad (43)$$

Since  $r_h$  is inversely proportional to the Hawking temperature  $T_H$ , we can see that absorption cross section is inversely related to Hawking temperature.

## 7. Hawking temperature via tunneling

Now we will determine the Hawking temperature using tunneling mechanism<sup>16</sup>. This mechanism has been used by many authors<sup>17,18,19</sup> for determining the Hawking radiation of black holes in Einstein gravity. Even though the complete properties of Hawking radiation can be obtained using quantum field theory in curved space-time, the tunneling mechanism gives a simple understandable picture. According to this picture the radiation arises by a process similar to electron-positron pair creation in a constant electric field. Using tunneling picture of black hole radiation we can have a direct semi-classical derivation of black hole radiation. There are two different schemes in tunneling approach, first is the radial null geodesic method and the other is Hamilton-Jacobi method. Here we use the radial null geodesic method. The metric in Eq.(5) can be transformed in to Painleve<sup>20</sup> like coordinate system to remove the singularity in the original extended metric as,

$$ds^2 = - \left( 1 - \frac{2\alpha m}{r^2} \right) dt^2 + 2\sqrt{\frac{2\alpha m}{r^2}} dt dr + dr^2 + r^2 d\Omega^2. \quad (44)$$

The radial null geodesics are

$$\frac{dr}{dt} = \dot{r} = \pm 1 - \sqrt{\frac{2\alpha m}{r^2}}. \quad (45)$$

The typical wavelength of the radiation is of the order of the size of the black hole and hence when the outgoing wave is traced back towards the horizon its wavelength as measured by local observers, is blue shifted. Near the horizon, radial wave number approaches to infinity and we can use WKB approximation to study for the particle tunneling . We start with original action

$$S = \int p(r) dr. \quad (46)$$

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Using Hamilton's equation of motion  $p(r) = \frac{dH}{dr}$  we can write the imaginary part of the action as

$$ImS = Im \int_{r_{in}}^{r_{out}} p_r dr = Im \int_m^{m-\omega} \int_{r_{in}}^{r_{out}} \frac{dr dH}{\dot{r}} \quad (47)$$

Using Eq.(45), the imaginary part of the action S can be written as

$$ImS = Im \int_0^\omega \int_{r_{in}}^{r_{out}} \frac{dr d(-\omega)}{\left(1 - \sqrt{\frac{2\alpha(m-\omega)}{r^2}}\right)}. \quad (48)$$

where  $\omega$  is the frequency of out going particle. Eq.(48) can be integrated to obtain

$$ImS = \frac{4\pi\sqrt{2\alpha}}{3} (m - \omega)^{\frac{3}{2}}. \quad (49)$$

Since  $m$  is much greater than  $\omega$  we can apply the binomial expansion to get

$$ImS = \frac{4\pi\sqrt{2\alpha}m}{3} \left(1 - \frac{3\omega}{2m}\right). \quad (50)$$

now the Semi-classical emission rate can be written as

$$\Gamma \sim e^{-2ImS} \sim \exp\left(\frac{4\pi\sqrt{2\alpha}m}{3} \left(\frac{3\omega}{2m}\right)\right) \quad (51)$$

from which we get

$$T_H = \frac{1}{2\pi\sqrt{2\alpha}m}. \quad (52)$$

There exist higher order corrections of  $\omega$  due to the conservation of energy, but for the first order calculation it is neglected. It is possible to find out the frequency dependent transmission coefficient or graybody factors in this tunneling scenario.

## 8. Conclusion

In this paper we have studied the scattering properties of extended black holes in F(R) theory. We have obtained the scattered wave equation in both regions near the horizon and away from horizon. Using the scattering method we have obtained the absorption cross section and also calculated the Hawking temperature of the black hole via tunneling method.

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